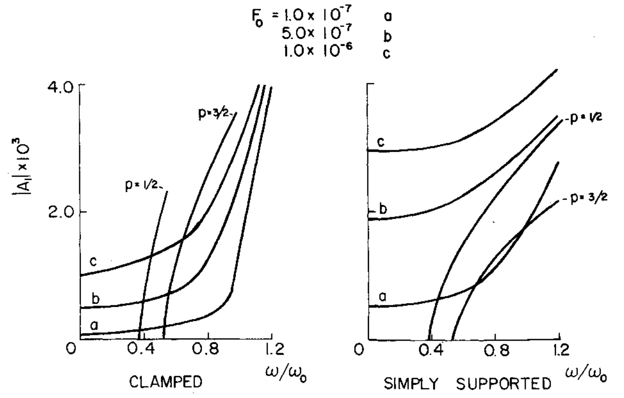

Fig. 2 Fundamental response and ultraharmonic vertical tangents for simply supported and clamped beams.

harmonic solutions will be followed until limited by damping. Now consider some point (B) on the curve $F_0 = 10^{-6}$. The bifurcation point (D) must be that given by the solutions to Eqs. (9); furthermore, when this point is reached, if the ultraharmonic is to occur the solution must jump to point (C). It is anticipated that unless a large perturbation occurs in the neighborhood of the bifurcation point that the ultraharmonic will not occur.

In order to substantiate this hypothesis Eq. (3) has been numerically integrated in the region just past the bifurcation point for $F_0 = 10^{-6}$ for several values of initial conditions. For values near the harmonic solution no tendency to jump to the lower branch of the curve was noted. The numerical integration has also confirmed the double valued aspects of the ultraharmonic response.

It is clear then that the location of the vertical tangent plays a significant part in determining the existence of the ultraharmonic response. If the vertical tangent of the $A_1 - \omega$ ultraharmonic solution lies below the fundamental response curve the ultraharmonic will be double valued and will probably not occur.

Figure 1 indicates that the vertical tangent may be adequately approximated by Eqs. (10). Rewriting the second of Eqs. (10) as

$$f = A_1(\omega/\omega_0)^2 - [A_1 + \tfrac{9}{4}GA_1{}^3 + P_0]/(2p^2 - 1)$$

The vertical tangency condition is

$$d(\omega/\omega_0)/dA_1 = 0 = -\partial f/\partial A_1/\partial f/\partial(\omega/\omega_0)$$

or

$$A_1{}^2 = [4(2p^2-1)/27G][1/(2p^2-1) - (\omega/\omega_0)^2] \tag{11}$$

As shown previously, the ultraharmonic will not exist for P_0 greater than the value for which the line of vertical tangents passes through the bifurcation point, since for any greater P_0 the vertical tangent will be below the fundamental curve. Furthermore, the lowest frequency for the bifurcation point is $\omega/\omega_0 = 1/p$. The intersection of these points will determine a maximum P_0 for an ultraharmonic to occur for increasing ω/ω_0. Equation 5 may be solved for P_0, and A_1 replaced by Eq. (11). If the resulting equation is evaluated at $\omega/\omega_0 = 1/p$ the following is obtained

$$P_0 = [4(1-p^2)/27p^2]^{1/2}[8(p^2-1)/9p^2]\, G^{-1/2} \tag{12}$$

Equation (12) will determine a maximum value of P_0 for an ultraharmonic to occur for increasing ω/ω_0.

The vertical tangent line (11) is quite dependent on the beam nonlinearity parameter G. The response curves (7) and the vertical tangent curves (11) have been plotted in Fig. 2 for several values of F_0 and for both simply supported and clamped boundary conditions, representing two extremes of support, again $h/L = 0.005$. If the vertical tangent lies below a response curve for a given F_0, in the region where the ultraharmonic can exist, the double valued ultraharmonic will exist. There is obviously a complex interplay among the various parameters which influence the behavior of the vertical tangent curve. It is clear however that any changes which increase the nonlinearity parameter G will flatten out the vertical tangent curve and lower the force level at which the double value phenomena will occur. The parameter G_1 which depends on integrals of the mode shape is less susceptible to change than the linear frequency ω_0. Therefore the ultraharmonic response can be changed most readily by increasing or decreasing the linear natural frequency. The boundary restraints play a significant role as shown in Fig. 2. For $F_0 = 5 \times 10^{-7}$ the $\frac{3}{2}$ ultraharmonic double valued ultraharmonic will occur for the simply supported boundary condition; whereas, it will not occur for the clamped boundary conditions.

3. Conclusions

It has been shown that for certain values of the parameters describing the geometry and forcing conditions of a nonlinear beam that the ultraharmonic response will change from a single valued to a double valued response curve. When the response curve is double valued it is concluded that for increasing ω/ω_0 the ultraharmonic response is extremely unlikely to occur. It has been demonstrated that for the double valued response the 3-term harmonic balance solution is necessary to predict the bifurcation point.

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A Consistent Treatment of Transverse Shear Deformations in Laminated Anisotropic Plates

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Introduction

RECENT interest in the analysis of the effect of transverse shear deformation in plates consisting of anisotropic laminations, and an apparent uncertainty about the appropriate way to extend results valid for homogeneous plates to the case of laminated plates[5] suggested to the writer that he should attempt to describe a way in which his earlier results for homogeneous isotropic plates,[1] and for sandwich plates with homogeneous cores,[2] can be generalized so as to apply to laminated anisotropic plates. For simplicity's sake this will be done in what follows for plates symmetric about their midplanes, without the previously considered possibility of coupled bending and stretching which exists without this symmetry.[4]

Our earlier results on the effect of transverse shear deformation were originally obtained through an application of the principle of minimum complementary energy in conjunction with the Lagrange multiplier method. It was subsequently shown that equivalent results could be obtained somewhat more simply

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through use of a variational theorem for stresses and displacements.[3] In what follows we will again employ the latter procedure.

Variational Equation

We consider a plate of thickness $2c$, with midplane coordinates x_1 and x_2 and with thickness coordinate z, the faces of the plate being at $z = \pm c$. We assume that the material of the plate is anisotropic in such a way that the complementary energy density of the plate (which here is equivalent to the strain energy density expressed in terms of stresses) is of the following form

$$W = (\sigma_i \sigma_j / 2E_{ij}^*) + (\tau_i \tau_j / 2G_{ij}^*) \tag{1}$$

In this σ_1 and σ_2 are normal stresses in the x_1 and x_2-directions, while σ_3 is the corresponding shear stress, which is usually designated by τ_{12}. At the same time τ_1 and τ_2 are transverse shear stresses, equivalent to what would usually be designated by τ_{1z} and τ_{2z}. It is assumed in stating (1) that the effect of transverse normal stress—which is here designated by τ_3—is negligible, by prescribing $G_{i3}^* = \infty$, and that the summation convention is used, with $E_{ij}^* = E_{ji}^*$ and $G_{ij}^* = G_{ji}^*$ being even functions of z.

Omitting for simplicity's sake the consideration of boundary terms, we then have a variational equation $\delta\Pi = 0$, for stresses and displacements, where

$$\Pi = \iint \int_{-c}^{c} (\varepsilon_i \sigma_i + \gamma_i \tau_i - W)\, dz\, dx_1\, dx_2 + \iint pw\, dx_1\, dx_2 \tag{2}$$

In this ε_1, ε_2, γ_3 are normal strains and ε_3, γ_1, γ_2 are shear strains, expressed in terms of appropriate derivatives of displacement components. The quantity p is the intensity of the transverse loads acting over the plate, and w the associated deflection.

The variational equation $\delta\Pi = 0$ will be used for the derivation of an approximate system of plate equations, on the basis of appropriate consistent approximate expressions for stresses and displacements.

Expressions for Stresses and Displacements

We approximate displacements in the same way as for homogeneous plates, that is by setting $u_1 = z\phi_1$, $u_2 = z\phi_2$, $u_z = w$ where ϕ_1, ϕ_2 and w are functions of x_1 and x_2 which remain to be determined. With this we have as approximations for the components of strain in Eq. (2),

$$\varepsilon_1 = z\phi_{1,1}, \quad \varepsilon_2 = z\phi_{2,2}, \quad \varepsilon_3 = z\phi_{1,2} + z\phi_{2,1} \tag{3}$$

$$\gamma_1 = \phi_1 + w_{,1}, \quad \gamma_2 = \phi_2 + w_{,2}, \quad \gamma_3 = 0 \tag{4}$$

We next approximate the middle surface parallel stresses σ_i in a manner compatible with Eq. (3) and with the form of the complementary energy density in (2) as

$$\sigma_i = E_{ij} \kappa_j z \tag{5}$$

In this the $E_{ij} = E_{ji}$ are given even functions of z, consistent with the functions E_{ij}^* in a manner which will be indicated, and the κ_j are functions of x_1 and x_2 which remain to be determined [and which are expected to come out to be such that $z\kappa_j = \varepsilon_j$, with ε_j as in (3)].

Having Eq. (5) we may now approximate the transverse shears τ_1 and τ_2 in various ways. In order to apply the principle of minimum complementary energy we would satisfy equilibrium by setting

$$\tau_1 = -\int_{-c}^{z} (\sigma_{1,1} + \sigma_{3,2})\, dz, \quad \tau_2 = -\int_{-c}^{z} (\sigma_{3,1} + \sigma_{2,2})\, dz \tag{6}$$

With this the approximating functions τ_1 and τ_2 come out to be linear combinations of the six quantities $\kappa_{j,1}$ and $\kappa_{j,2}$

$$-\tau_1 = \kappa_{j,1} \int_{-c}^{z} E_{1j} z\, dz + \kappa_{j,2} \int_{-c}^{z} E_{3j} z\, dz \tag{7a}$$

$$-\tau_2 = \kappa_{j,1} \int_{-c}^{z} E_{3j} z\, dz + \kappa_{j,2} \int_{-c}^{z} E_{2j} z\, dz \tag{7b}$$

If we do not apply the principle of minimum energy but instead apply the variational principle for stresses and displacements we have more freedom in choosing expressions for τ_1 and τ_2. We may, for example, elect most simply to set $\tau_i = Q_i/2c$ [which is compatible with Eq. (6) for the class of sandwich-type plates considered in Ref. 2]. Alternately we may set $\tau_i = (3/2)(Q_i/2c)(1 - z^2/c^2)$ [and this is compatible with Eq. (6) for the class of isotropic homogeneous plates considered in Ref. 1]. In what follows we decide to simplify expressions (7a, b) in a manner which is flexible enough to allow the possibility that the expressions for τ_i *may* assume the form (7a, b), in the event that the approximate variational equation would wish to lead to this consequence, by setting with six functions $\lambda_k(x_1, x_2)$ which remain to be determined,

$$\tau_i = -A_{ik} \lambda_k; \quad i = 1, 2, \quad k = 1, \ldots, 6 \tag{8}$$

where

$$A_{11} = \int_{-c}^{z} E_{11} z\, dz, \; A_{12} = \int_{-c}^{z} E_{12} z\, dz, \; A_{21} = \int_{-c}^{z} E_{31} z\, dz \tag{9}$$

etc. Since $\gamma_3 = 0$, and τ_3 does not occur in W, it is unnecessary to consider the form of an approximation for τ_3.

Having Eqs. (3) to (5) together with (8) and (9) we may now use the variational equations $\delta\Pi = 0$ for the sake of a derivation of two-dimensional plate equations which include the customary equilibrium equations for stress couples and resultants, as well as constitutive equations, with the form of the latter being the principal object of our considerations.

Approximate Variational Equation

We evaluate the various integrals with respect to z in the expression for Π, as follows

$$\int_{-c}^{c} \varepsilon_i \sigma_i\, dz = \phi_{1,1} M_1 + \phi_{2,2} M_2 + (\phi_{1,2} + \phi_{2,1}) M_3 \tag{10}$$

$$\int_{-c}^{c} \gamma_i \tau_i\, dz = (\phi_1 + w_{,1}) Q_1 + (\phi_2 + w_{,2}) Q_2 \tag{11}$$

where, by way of definition,

$$M_i = D_{ij} \kappa_j, \quad Q_i = C_{ik} \lambda_k \tag{12}$$

and

$$D_{ij} = \int_{-c}^{c} z^2 E_{ij}\, dz, \quad C_{ik} = -\int_{-c}^{c} A_{ik}\, dz \tag{13}$$

Next,

$$\int_{-c}^{c} \frac{\sigma_i \sigma_j}{E_{ij}^*}\, dz = D_{kl} \kappa_k \kappa_l \tag{14†}$$

and

$$\int_{-c}^{c} \frac{\tau_i \tau_j}{G_{ij}^*}\, dz = B_{kl} \lambda_k \lambda_l \tag{15}$$

with the quantities B_{kl} defined as

$$B_{kl} = \int_{-c}^{c} (A_{ik} A_{jl} / G_{ij}^*)\, dz \tag{16}$$

Introduction of (10, 11, 14, and 15) into Eq. (2) then gives the approximate variational equation

$$\delta \iint \{\phi_{1,1} M_1 + \phi_{2,2} M_2 + (\phi_{1,2} + \phi_{2,1}) M_3 + (\phi_i + w_{,i}) Q_i - \tfrac{1}{2} D_{kl} \kappa_k \kappa_l - \tfrac{1}{2} B_{kl} \lambda_k \lambda_l + pw\}\, dx_1\, dx_2 = 0 \tag{17}$$

with M_i, and Q_i defined in terms of κ_j and λ_k by means of Eq. (12), and with $\delta\phi_i$, δw, $\delta\kappa_j$, and $\delta\lambda_k$ being altogether twelve independent variations.

It is then evident that the Euler equations coming from $\delta\phi_i$ and δw are the usual equilibrium equations for transverse bending of plates, and the associated constitutive equations must come from the terms with $\delta\kappa_j$ and $\delta\lambda_k$, in conjunction with the equations of definition (12).

† Here we have made use of the fact that, necessarily, $E_{ik} E_{jl} / E_{ij}^* = E_{kl}$.

The relevant results are obtained as follows. Varying κ_j we obtain three Euler equations of the form

$$\phi_{1,1} D_{1i} + \phi_{2,2} D_{2i} + (\phi_{1,2} + \phi_{2,1}) D_{3i} = D_{1i} \kappa_1 + D_{2i} \kappa_2 + D_{3i} \kappa_3 \qquad (18)$$

and these are equivalent to the two-dimensional strain displacement relations

$$\kappa_1 = \phi_{1,1}, \quad \kappa_2 = \phi_{2,2}, \quad \kappa_3 = \phi_{1,2} + \phi_{2,1} \qquad (19)$$

which, upon introduction into (12), result in the set of constitutive equations for the M_i.

Next, varying λ_k we obtain six Euler equations of the form

$$(\phi_i + w_{,i}) C_{ik} = B_{kl} \lambda_l \qquad (20)$$

We solve Eqs. (20) for the λ_l, giving us six strain displacement relations

$$\lambda_l = F_{il} (\phi_i + w_{,i}) \qquad (21)$$

with the coefficients F_{il} being appropriate combinations of the coefficients C_{ik} and B_{kl}.

Finally, introduction of (21) into the expression for Q_i in Eq. (12) gives as constitutive equations for the Q_i,

$$Q_i = H_{ij} (\phi_j + w_{,j}) \qquad (22)$$

where $H_{ij} = C_{ik} F_{jk}$. Although Eq. (22) has been the principal object of our considerations, we note that, beyond this, we have in Eqs. (8) and (21) a means of determining approximate values of the transverse shear stresses themselves. We further note that exact considerations of some specific cases indicate that approaches such as the present one, leading to a two-dimensional sixth-order plate theory, in general are useful only if the transverse shear coefficients G are not *too* small in comparison with the in-plane moduli E. If L is a representative length for significant changes of stress and strain along directions in the x_1, x_2 plane then ratios G/E as small as c/L come out to be alright, but ratios G/E as small as c^2/L^2 are no longer so.

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